Pricing Foreign Exchange Options

Incorporating Purchasing Power Parity

Second edition

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CHAPTER 1
PREAMBLE

This book is a revised and re-written version of part of a study, sponsored (from 1992 onward) by Cypress International Investment Advisors Ltd. Its purpose is to describe a new approach to the valuation of options on foreign exchange. Though the core of the original text remains, comments, especially from professional practitioners, have led the authors to re-orientate the exposition. In particular, the present edition has been recast with *applications* very much in mind.

The reason underlying this change in exposition would be clear if one remembers a bit of methodology. According to Friedman's well known view (1953), in positive economics assumptions do not matter. When one's purpose is to test a theory, what is important is that its predictions are not empirically falsified. However, in *applied* economics the situation is different, for in this case assumptions matter very much indeed. When one is applying a theory to study economic growth in Hong Kong and produce policy recommendations, it would not do if it assumes a closed economy or an infinitely elastic supply of land.

The same may be said about option pricing, which is essentially an application of capital theory. In this case, as Cox & Ross (1976) have shown, what one assumes about the stochastic specifications governing the price of the underlying asset is of fundamental importance. The use of stochastic processes[1] to model the price of assets was pioneered by Bachelier (1900). The idea was that in a continuous competitive market, the asset price would be subject to so many independent influences that we can imagine it to fluctuate randomly along a continuous path. As a result, Bachelier assumed the price of the representative asset to follow (what is now called) a Brownian motion.

Brownian motion, however, allows asset prices to go negative. Since this would violate the condition of limited liability, the assumption cannot be used when applying the analysis to an equity. (It would also be difficult to

[1] Technical terms are explained in Chapter 3 below.

apply to any good which has a positive price and therefore a market). This result led Samuelson (1965) to introduce the idea of geometric Brownian motion, in which the asset price is restricted to take positive values with non-zero probabilities. Since that time, geometric Brownian motion has become a paradigm for financial research. In the particular subject of options, the seminal work of Black & Scholes (1973) is based on the assumption that the representative stock price follows geometric Brownian motion.

Recent research (to which the authors contributed[2]) has uncovered a number of problems, which suggests that the scope of application of geometric Brownian motion is not as wide as first envisaged. Institutionally, it is clear that the assumption cannot be applied to the bond market, and to value fixed income options and the options embedded in callable bonds. Since geometric Brownian motion allows asset prices to go infinite with non-zero probability but every bond has a finite maximum price, as long as interest rates are non-negative it would be inappropriate to model the representative bond price in such terms (Dyer & Jacob 1996). In addition, there are serious problems in theory. For example, if an asset price follows geometric Brownian motion, it is possible for its sample path to drop to 0 with probability 1, and yet all the time the expected price of the asset would be increasing without limit. An individual who holds such an asset according to standard portfolio (mean-variance) critieria would be "almost certainly ruined", with a zero price the market would disappear, and options on the asset would yield distorted values.

In this book we propose an alternative assumption to geometric Brownian motion, and show how it can be applied to perhaps the largest financial market in the world, that for foreign exchange. This new stochastic specification is free from the theoretical problems noted above. "Almost certain ruin" and the disappearance of markets are excluded for a representative foreign currency. In addition, the non-random effects of standard economic theory (in particular changes in purchasing power parity) can be incorporated, both in the description of the stochastic process for the spot price of the currency and in a new formula for pricing foreign exchange options.

[2] See Cheung & Yeung (1994a) and (1994b).

Chapter 2 introduces definitions and terminology. To save on time spent looking up textbooks, a technical glossary is supplied in Chapter 3. Chapter 4 attempts to impress upon the reader the importance of assumptions in option theory, by presenting an example in which an option price is obtained without making stochastic assumptions at all, and inviting the reader to compare it with the classic Black-Scholes formula. Black and Scholes contribution (1973), which is fundamental to all modern work in options, is discussed in Chapters 5 and 6. Since (as noted above) the underlying asset price is assumed to follow geometric Brownian motion, two serious problems are seen to arise. First, the technique commonly used to solve Black-Scholes differential equations does not exclude "almost certain ruin", so that it is difficult to maintain the required general equilibrium interpretation of the resulting option prices. Secondly, under geometric Brownian motion the asset price displays the characteristics of a random walk, in the sense that its value at any future point of time depends solely on what it is at present. This property is beginning to be called into question by recent research. (See e.g. McQueen & Thorley 1991, Samuelson 1991, Kaehler & Kugler eds. 1994, Haugen 1995, Malkiel 1996, Campbell, Lo & Mackinlay 1997). For example, it is suggested that returns to U.S. common stocks in the post-war period show statistically significant non-random walk behavior, especially that runs of high and low returns have been found to follow one another.

To meet the problems which arise from assuming geometric Brownian motion, we propose (in Chapter 7) an alternative stochastic process to model the dynamic behavior of asset prices. The solution of the resulting stochastic differential equation is characterised completely, in the form of a closed form expression for the asset price's transition density function. It is shown that "almost certain ruin" is excluded, and non-random walk effects from standard economic theory — for example, the changes in the representative firm's equilibrium balance sheet which underlie the Modigliani-Miller Theorem — can be taken into account. An option pricing formula is also obtained, by taking mathematical expectation in terms of the asset price transition density. Chapter 8 shows how the stochastic specification of Chapter 7 can be applied to model the spot price of a representative foreign currency, and to price options on it. In contrast to the random walk restriction imposed by geometric Brownian motion, we are able to incorporate a fundamental theorem of international finance, that in the long run the exchange rate converges to purchasing power parity. "Almost certain ruin", the disappearance of markets and their consequences are excluded, and finally a computable closed-form formula to price foreign exchange options is obtained.

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CHAPTER 3
TECHNICAL GLOSSARY

3.1 Introduction[1]

The technical terms and results which will be used in the exposition are summarised in this Chapter. For details and proofs, the reader is referred to any good text on stochastic processes, e.g., Karlin & Taylor (1975, 1981).

3.2 Stochastic Processes

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and T an arbitrary set of numbers. Suppose we define the function:

$$X(t, \omega), \quad t \in T, \ \omega \in \Omega. \tag{3.1}$$

A stochastic process is a family $\{X(t, \omega)\}$ of such functions. For any given $t \in T$, $X(t, \cdot)$ denotes a random variable (or a random vector) on the probability space $(\Omega, \mathcal{A}, P)$. For any fixed $\omega \in \Omega$, $X(\cdot, \omega)$ is a real valued function (vector valued function) defined on T, called a sample path or realisation of the stochastic process. The standard notation suppresses the variable ω, so that a stochastic process is written $\{X(t)\}$.

The theory of stochastic processes is concerned with the structure and properties of $\{X(t, \omega)\}$ under different assumptions. The main elements which distinguish stochastic processes are the state space S, which is the set of values the random variable $X(t, \cdot)$ may take, the index set T, and the dependence relationships among the random variables $X(t, \cdot)$.

If the state space $S = \{0, 1, 2, \cdots\}$, the stochastic process $\{X(t)\}$ is described as integer valued. If S is the real line $(-\infty, \infty)$, $\{X(t)\}$ is a real valued stochastic process. If S is a k dimensional Euclidean space, $\{X(t)\}$ is a k-vector stochastic process.

If the index set $T = \{0, 1, 2, \cdots\}$, $\{X(t)\}$ is a discrete stochastic process. If $T = (-\infty, \infty)$, the stochastic process $\{X(t)\}$ is continuous. Often, the variable t is interpreted to be time. Then, if $T = (-\infty, \infty)$, $\{X(t)\}$ would be a continuous time stochastic process.

[1] The notation is special to this Chapter.

Different dependence relationships among the $X(t)$ give rise to different stochastic processes. If we wish to characterise the stochastic process $\{X(t)\}$, this requires the knowledge of (countably or uncountably many, depending on the nature of T) joint distributions of the random variables (or random vectors) $X(t)$. The set of all such joint distributions:

$$\begin{aligned} &F_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1,x_2,\cdots,x_n) \\ &= [\Pr X(t_1) \le x_1,\ X(t_2) \le x_2,\ \cdots,\ X(t_n) \le x_n], \end{aligned}$$

for all $t_1,t_2,\cdots,t_n \in T$, $t_i \neq t_j$, $i \neq j$, constitutes the probability law of the stochastic process.

Generally, the random variables $X(t)$ are interdependent. If, for all choices of $t_1,\cdots,t_n$, $t_i \in T$ such that

$$t_1 < t_2 < \cdots < t_n$$

the random variables

$$X(t_2) - X(t_1),\ X(t_3) - X(t_2), \cdots,\ X(t_n) - X(t_{n-1})$$

are independent, then $\{X(t)\}$ is a stochastic process with independent increments. If the index set T contains a smallest element t_0, it is assumed that the random variables $X(t_0)$, $X(t_1) - X(t_0)$, $\cdots$, $X(t_n) - X(t_{n-1})$ are independent.

If the index set $T = \{0,1,2,\cdots\}$, then a stochastic process with independent increments reduces to a sequences of independent random variables $Z(0) = X(0)$, $Z(i) = X(i) - X(i-1)$, $i = 1,2,\cdots,n$.

If (for any t) the distribution of the random variables $X(t+h) - X(t)$ depends only on the length h of the interval and not on t, the stochastic process $\{X(t)\}$ is said to possess stationary increments. Given a stochastic process with stationary increments, the distribution of $X(t_1+h) - X(t_1)$ is the same as the distribution of $X(t_2+h) - X(t_2)$, no matter what the values of t_1, t_2 and h.

A stochastic process $\{X(t)\}$ is said to be strictly stationary if the joint distribution functions of the two sets of random variables:

$$\{X(t_1+h),\ X(t_2+h),\ \cdots,\ X(t_n+h)\},\ \{X(t_1), X(t_2), \cdots, X(t_n)\}$$

are the same for all $h > 0$ and all choices of $\{t_1, t_2, \cdots, t_n\}$ from T. This condition says that the stochastic process is in probabilistic equilibrium, so that the specific instances at which we examine the process are irrelevant. In particular, the distribution of $X(t)$ is the same for each $t \in T$.

If the stochastic process $\{X(t)\}$ possesses finite second moments and if $\text{cov}[X(t), X(t+h)]$ depends only on h for all $t \in T$, it is said to be wide sense stationary. A stationary stochastic process with finite second moments is wide sense stationary, but there are wide sense stationary stochastic processes which are not stationary. In economics, stationary stochastic processes are frequently used in rational expectations models, to characterise stochastic equilibrium (in the macroeconomic sense).

3.3 Martingales

Let $\{X(t)\}$ be a real valued stochastic process with a discrete parameter set T. Then it is a martingale if:

(a) $E\big[|X(t)|\big] < \infty, \ \forall\, t \in T$,

(b) $E\big[X(t_{n+1})\big|X(t_1) = a_1,\ X(t_2) = a_2, \cdots,\ X(t_n) = a_n\big] = a_n$, for any $t_1 < t_2 < \cdots < t_n < t_{n+1},\ t_i \in T$.

More generally, if $\{X(t)\}$ and $\{Y(t)\}$ are stochastic processes with the discrete parameter set T, $X\{(t)\}$ is a martingale with respect to $\{Y(t)\}$, if:

(a) $E\big[|X(t)|\big] < \infty, \ \forall\, t \in T$,

(b) $E\big[X(t_{n+1})\big|Y(t_1) = b_1,\ Y(t_2) = b_2, \cdots,\ Y(t_n) = b_n\big] = b_n$, for any $t_1 < t_2 < \cdots < t_n < t_{n+1},\ t_i \in T$.

Martingales are considered to be appropriate models for fair games, in which the random variable $X(t)$ represents the amount of money a player possesses at time t. The martingale property states that the average amount the player would have at time t_{n+1}, given that he has a_n at time t_n, is equal to a_n regardless of what his past fortune has been.

3.4 Markov Stochastic Processes

A Markov stochastic process has the property that, given the value of

$X(s)$, the values of $X(t)$, $t > s$, do not depend on the values of $X(u)$, $u < s$. That is, the probability of any particular future behavior of the process, when its present state is known exactly, is not changed by knowledge about its past behavior.

More formally, if for any

$$t_1 < t_2 < \cdots < t_n < t,$$
$$\Pr\left[a < X(t) \leq b | X(t_1) = x_1,\ X(t_2) = x_2, \cdots,\ X(t_n) = x_n\right]$$
$$= \Pr\left[a < X(t) \leq b | X(t_n) = x_n\right],$$

then $\{X(t)\}$ is a Markov stochastic process.

Suppose $T = (-\infty, \infty)$, and $A = (a, b]$ is an interval of the real line. The function

$$P(x, s; t, A) = \Pr\left[X(t) \in A | X(s) = x\right], \quad t > s,$$

is called the transition probability function of the Markov stochastic process $\{X(t)\}$. In particular, it can be proved that the probability distribution of the set of random variables $\{X(t_1), X(t_2), \cdots, X(t_n)\}$ (the probability law of the stochastic process) can be found in terms of the transition probability function of the process and the initial distribution function of $X(t)$.

A Markov stochastic process $\{X(t)\}$ with a finite or countably infinite state space $S = \{0, 1, 2, \cdots, n\}$ or $S = \{0, 1, 2, \cdots\}$ is called a Markov chain. A Markov stochastic process $\{X(t)\}$ for which all sample functions $\{X(t, \omega),\ t \in T = (-\infty, 0]\}$ are continuous in t is called a diffusion process. Under certain conditions, the transition probability function $P(x, s; t, \cdot)$ has a transition density function $p(x, s; t, \cdot)$. A Markov process is said to possess stationary transition probabilities if the transition probability function $P(x, s; t, \cdot)$ (and the transition density function $p(x, s; t, \cdot)$, if it exists) is a function only of $(t - s)$. Notice that a stochastic process with stationary transition probabilities is not necessarily stationary.

3.5 Random Walks

A discrete time Markov chain is a Markov stochastic process with a finite or countably infinite state space $S = \{0, 1, 2, \cdots\}$, and index set $T =$

$\{0, 1, 2, \cdots\}$. It is common to write a Markov chain as $\{X(n)\}$ or $\{X_n\}$ instead of $\{X(t)\}$, and to say that X_n is in state i if $X_n = i$.

The probability of X_{n+1} being in state j, given that X_n is in state i, called a one-step transition probability, is denoted by:

$$P_{ij}^{n,n+1} = \Pr\big(X_{n+1} = j | X_n = i\big).$$

The notation emphasises that in general, the transition probabilities of the Markov chain depend on the initial state i and final state j, and on the time interval over which the transition occurs $(n, n+1)$. If one-step transition probabilities are independent of the time of transition (n), then (as we have seen) the Markov chain possess stationary transition probabilities.

In this case,

$$P_{ij}^{n,n+1} = P_{ij}.$$

Since P_{ij} is a probability,

$$P_{ij} \geq 0, \ i, j = 0, 1, 2, \cdots, \quad \sum_{j=0}^{\infty} P_{ij} = 1, \ i = 0, 1, 2, \cdots$$

(The summation condition expresses the fact that some transition occurs in each step, or each trial, of the process.) It can be shown that the Markov chain is completely determined once P_{ij} is known for all i and j, and the probability distribution of X_0 is specified.

A random walk is a Markov chain in which X_n, if it is in state i, can in a single transition either remain in state i, or move to one of the adjacent states $i+1$ or $i-1$. In this case:

$$\begin{aligned}
\Pr\big(X_{n+1} = i+1 | X_n = i\big) &= p_i, \\
\Pr\big(X_{n+1} = i-1 | X_n = i\big) &= q_i, \\
\Pr\big(X_{n+1} = i | X_n = i\big) &= r_i,
\end{aligned}$$

where $p_i > 0, \ q_i > 0, \ r_i \geq 0, \ p_i + q_i + r_i = 1, \ i = 1, 2, \cdots, \ p_0 \geq 0, \ r_0 \geq 0, \ p_0 + r_0 = 1$. If $p_i = q_i = p \geq 0$ and $r_i = r \geq 0$, the random walk is symmetric.

The fortune of an individual in a game is often depicted by a random walk stochastic process. Suppose the player has fortune k and plays a game against an (infinitely rich) adversary, with the probability p_k of winning one dollar and probability $q_k = 1 - p_k$ of losing one dollar in each trial, and $r_0 = 1$. If the random variable X_n represents the individual's fortune after n trials, the stochastic process $\{X_n\}$ is a random walk, known as gambler's ruin. (Once state 0 is reached, the process will remain in it.)

3.6 Brownian Motion

The study of Brownian motion began with the observation by the Scottish botanist R. Brown in 1827, that small particles like pollen grains immersed in a liquid exhibit ceaseless irregular motions. In 1905, Einstein explained this phenomenon by a theory in which the particles under observation are subject to perpetual collisions with the molecules of the surrounding medium. Einstein's results were later extended by various physicists and mathematicians, for example N. Wiener and S. Chandrasekhar. (Brownian motion is also known as a Wiener stochastic process.)

At time $t \in T = (-\infty, 0]$, let $X(t)$ denote the displacement (from a starting point along a fixed axis) of a Brownian particle. The displacement $X(t) - X(s)$ over the time interval (s, t) can be regarded as the sum of a large number of small displacements. The central limit theorem is then applicable, so we can assert that the random variable $X(t) - X(s)$ is normally distributed. It is intuitively clear that the displacement $X(t) - X(s)$ depends only on $(t - s)$ and not on the time we begin the observation. Moreover, it is reasonable to assume that the Brownian motion is in stochastic equilibrium, in the sense that the distribution of $X(t+h) - X(s+h)$ is the same as the distribution of $X(t) - X(s)$, for all $h > 0$.

Given these observations, the Brownian motion stochastic process $\{X(t), t \geq 0\}$ possesses the following characteristics:

(a) Given $t_0 < t_1 \cdots < t_n$, the increments $X(t_1) - X(t_0), \cdots, X(t_n) - X(t_{n-1})$ are (mutually) independent random variables;

(b) the probability distribution function of $X(t) - X(s)$, $t > s$, depends only on $(t-s)$ and not on t or s;

(c) $\Pr[X(t)-X(s) \le x] = 1/\sigma\sqrt{[2\pi(t-s)]}\int_{-\infty}^{x} \exp\{-u^2/[2\sigma^2(t-s)]\}du$, where $t > s$ and σ is a positive constant.

Assume that for each sample path of the process, $X(0) = 0$. It can then be proved that, conditional upon $X(0) = 0$,

$$E[X(t)] = 0, \quad \mathrm{var}[X(t)] = \sigma^2 t,$$

and that for $0 < t_1 < t_2 < \cdots < t_n < t$, the conditional probability distribution of $X(t)$ given $X(t_1)$, $X(t_2)$, $\cdots$, $X(t_n)$ is

$$\Pr[X(t) \le x | X(t_1) = x_1,\ X(t_2) = x_2, \cdots,\ X(t_n) = x_n]$$
$$= \frac{1}{\sigma\sqrt{[2\pi(t-t_n)]}}\int_{-\infty}^{x-x_n} \exp\left\{-\frac{u^2}{2\sigma^2(t-t_n)}\right\} du.$$

A discrete approximation to Brownian motion is provided by a symmetric random walk.

3.7 Geometric Brownian Motion

Let $\{X(t),\ t \in [0,\infty)\}$ be a Brownian motion stochastic process. Brownian motion with drift is a stochastic process $\{U(t),\ t \in [0,\infty)\}$, where

$$U(t) = X(t) + \mu t,$$

and the drift parameter μ is a constant.

Alternatively, we can define a Brownian motion with drift to be a stochastic process $\{U(t),\ t \in [0,\infty)\}$ with the properties:

(a) the increments $X(t+s) - X(s)$ are normally distributed with mean μ and variance $\sigma^2 t$, where μ and $\sigma > 0$ are constants;

(b) for every $t_1 < t_2 < \cdots < t_n$, the increments $X(t_2)-X(t_1), \cdots, X(t_n)-X(t_{n-1})$ are independent random variables with distributions given in (a);

(c) for every sample path, $X(0) = 0$, and $X(t,\omega)$ is continuous at $t = 0$.

Let $\{U(t),\ t \in [0,\infty)\}$ be a Brownian motion stochastic process with drift coefficient μ. The stochastic process defined by:

$$Y(t) = \exp[U(t)], \quad t \geq 0,$$

is called geometric Brownian motion. (The state space of the process is $(0,\infty)$.) It can be shown that the random variable $Y(t)$ is lognormally distributed with mean and variance:

$$E[Y(t)|Y(0) = Y_0] = Y_0 \exp\left[\mu t + \frac{1}{2}\sigma^2 t\right], \quad \text{and}$$
$$\text{var}[Y(t)|Y(0) = Y_0] = Y_0^2 \left[\exp(2\mu t + \sigma^2 t)\right]\left[\exp(\sigma^{2t} - 1\right].$$

That is, $Y(t)$ has the probability density function:

$$f(y) = \frac{1}{y\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(\log y - \mu t)^2}{2\sigma^2 t}\right\}, \quad y > 0.$$

3.8 Formulae from Stochastic Calculus

Consider a probability space $(\Omega, \mathcal{A}, P)$ and a stochastic process $\{X(t,\omega)\}$, $t \in [0,T]$. Let $\sigma(t,\omega)$ be a non-anticipating function and $f(t,\omega)$ be another function (both defined by properties which we can take for granted here). Then the stochastic process $\{X(t,\omega)\}$ or $\{X(t)\}$ has a stochastic differential denoted by:

$$dX(t) = f(t)dt + \sigma(t)dz(t),$$

where (see §3.6 above) $\{dz(t)\}$ is a Wiener (Brownian motion) stochastic process with $E[dz(t)] = 0$, $\text{var}[dz(t)] = dt$.

Let $u(t, X)$ be a continuous non-random function with continuous partial

derivatives. Then Itô's lemma states: if the stochastic process $\{Y(t,\omega)\}$ or $\{Y(t)\}$ is such that we have $Y(t) = u(t, X(t))$, it also possesses a stochastic differential given by:

$$dY(t) = \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial X} f(t) + \frac{1}{2}\sigma(t)^2 \frac{\partial^2 u}{\partial X^2}\right] dt + \frac{\partial u}{\partial X}\sigma(t)dz(t).$$

As we will see below, Itô's lemma is fundamental to Black and Scholes' theory of option pricing.

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CHAPTER 9
CONCLUSIONS

The appropriate choice of assumptions is a matter of primary importance in applied economics. In option pricing, it is generally assumed that the underlying asset has a price which fluctuates over time as a geometric Brownian motion. Since a number of problems then arise, the present volume proposes an approach to the valuation of foreign exchange options based on an alternative assumption. Definitions and terminology are introduced in Chapter 2. A technical glossary is given in Chapter 3. The remainder of the book is devoted to a new stochastic specification to model the spot price of the representative currency underlying foreign exchange options, and to showing that it is free from the problems which arise under the standard geometric Brownian motion assumption. An example is presented in Chapter 4 to convince the reader of the importance of stochastic assumptions in option pricing, and-to demonstrate how a different specification leads in general to a different valuation formula. Chapter 5 surveys the Black-Scholes (1973) theory, which is the basis of modern work on option pricing. It is shown in Chapters 6 and 7 that two serious problems arise in this theory. First, since it is assumed that the price of the underlying asset follows geometric Brownian motion, it is possible for assets to exist which lead the investor to "almost certain ruin" (Samuelson 1965), in the sense that over time their prices would drop to zero with probability one, positive and increasing expected rates of return notwithstanding. The technique commonly used to solve Black-Scholes differential equations for option prices does not exclude such assets, which makes it difficult to maintain an (general) equilibrium interpretation of the resulting option price. Second, given the geometric Brownian motion assumption, the price of an asset displays the characteristics of a random walk. Its price at any point of time in the future depends solely on the present price. This property has been called into question by recent research. For example, it is shown that returns to U.S. common stocks in the postwar period show significant non-random walk behavior, in the sense that runs of high and low returns tend to follow one another.

To overcome the difficulties arising from the assumption of geometric Brownian motion, Chapter 7 proposes an alternative stochastic specification to model the dynamic behavior of asset prices, in particular the price of the asset which underlies the representative option. We characterise the solution of the resulting system of stochastic differential equations in the most complete form known, by obtaining a closed form expression for the transition density function of the asset price. It is shown that non random walk effects can be incorporated into the analysis, and that the stochastic process of the asset price excludes the possibility of almost certain ruin. Since "the option valuation problem is equivalent to the problem of determining the distribution of the asset price", we proceed to take mathematical expectations directly in terms of the transition density function, to illustrate how an exact formula can be found to evaluate options on the asset.

Chapter 8 shows in detail how the new stochastic specification can be used to price options on foreign exchange. In contrast to the random walk restriction imposed by the standard theory, we are able to incorporate a fundamental result of the theory of international trade, that over time the spot price of a currency (its exchange rate) converges to its purchasing power parity. It is also shown that the exchange rate process excludes the possibility of almost certain ruin. Computational aspects of the formula are also discussed, in particular, the availability of data, and the methods by which the parameters in the option price formula can be estimated.

A Note on Software

A software package to compute the prices of foreign exchange options using formula (7.15) is being developed by the authors, under sponsorship from Cypress International Investment Advisors Ltd.

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